Package ‘weyl’

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Type Package

Title The Weyl Algebra

Version 0.0-4

Depends methods, R (>= 3.5.0)

Maintainer Robin K. S. Hankin <hankin.robin@gmail.com>

Description A suite of routines for Weyl algebras. Notation follows Coutinho (1995, ISBN 0-521-55119-6, ``A Primer of Algebraic D-Modules''). Uses 'disordR' discipline (Hankin 2022 <doi:10.48550/ARXIV.2210.03856>). To cite the package in publications, use Hankin 2022 <doi:10.48550/ARXIV.2212.09230>.

License GPL (>= 2)

LazyData yes

Suggests knitr,rmarkdown,testthat

VignetteBuilder knitr

Imports mathjaxr, disordR (>= 0.0-8), freealg (>= 1.0-4), spray (>= 1.0-19)

URL https://github.com/RobinHankin/weyl

BugReports https://github.com/RobinHankin/weyl/issues

RdMacros mathjaxr

R topics documented:

`weyl-package` *The Weyl Algebra*

Description

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Details

The DESCRIPTION file:

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Title:	The Weyl Algebra
Version:	0.0-4
Depends:	methods, R (>= 3.5.0)
Authors@R:	person(given=c("Robin", "K. S."), family="Hankin", role = c("aut","cre"), email="hankin.robin@gmai
Maintainer:	Robin K. S. Hankin <hankin.robin@gmail.com>
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VignetteBuilder:	knitr
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RdMacros:	mathjaxr
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zero	The zero operator

Author(s)

NA

Maintainer: Robin K. S. Hankin <hankin.robin@gmail.com>

Examples

```
x <- rweyl(d=1)
y <- rweyl(d=1)
z <- rweyl(d=1)

is.zero(x*(y*z) - (x*y)*z)  # should be TRUE
```

`coeffs` *Manipulate the coefficients of a weyl object*

Description

Manipulate the coefficients of a weyl object. The coefficients are `disord` objects.

Usage

```
coeffs(S) <- value
```

Arguments

S	A weyl object
value	Numeric

Details

To access coefficients of a weyl object `S`, use `spray::coeffs(S)` [package idiom is `coeffs(S)`]. Similarly to access the index matrix use `index(s)`.

The replacement method is package-specific; use `coeffs(S) <-value`.

Value

Extraction methods return a `disord` object (possibly `dropped`); replacement methods return a `weyl` object.

Author(s)

Robin K. S. Hankin

Examples

```
(a <- rweyl(9))
coeffs(a)
coeffs(a)[coeffs(a)<3] <- 100
a
```

`constant` *The constant term*

Description

The *constant* of a `weyl` object is the coefficient of the term with all zeros.

Usage

```
constant(x, drop = TRUE)
constant(x) <-  value
```

Arguments

`x`	Object of class `weyl`
`drop`	Boolean with default `TRUE` meaning to return the value of the coefficient, and `FALSE` meaning to return the corresponding Weyl object
`value`	Constant value to replace existing one

Value

Returns a numeric or weyl object

Note

The `constant.weyl()` function is somewhat awkward because it has to deal with the difficult case where the constant is zero and `drop=FALSE`.

Author(s)

Robin K. S. Hankin

Examples

```
(a <- rweyl()+700)
constant(a)
constant(a,drop=FALSE)

constant(a) <- 0
constant(a)
constant(a,drop=FALSE)

constant(a+66) == constant(a) + 66
```

`degree` *The degree of a* `weyl` *object*

Description

The *degree* of a monomial weyl object $x^a\partial^b$ is defined as $a+b$. The degree of a general weyl object expressed as a linear combination of monomials is the maximum of the degrees of these monomials. Following Coutinho we have:

- $\deg(d_1 + d_2) \leq \max(\deg(d_1) + \deg(d_2))$
- $\deg(d_1 d_2) = \deg(d_1) + \deg(d_2)$
- $\deg(d_1 d_2 - d_2 d_1) \leq \deg(d_1) + \deg(d_2) - 2$

Usage

```
deg(S)
```

Arguments

`S`	Object of class `weyl`

Value

Nonnegative integer (or $-\infty$ for the zero Weyl object)

Note

The degree of the zero object is conventionally $-\infty$.

Author(s)

Robin K. S. Hankin

Examples

```
(a <- rweyl())
deg(a)

d1 <- rweyl(n=2)
d2 <- rweyl(n=2)

deg(d1+d2) <= deg(d1) + deg(d2)
deg(d1*d2) == deg(d1) + deg(d2)
deg(d1*d2-d2*d1) <= deg(d1) + deg(d2) -2
```

`derivation` *Derivations*

Description

A *derivation* D of an algebra A is a linear operator that satisfies $D(d_1 d_2) = d_1 D(d_2) + D(d_1) d_2$, for every $d_1, d_2 \in A$. If a derivation is of the form $D(d) = [d, f] = df - fd$ for some fixed $f \in A$, we say that D is an *inner* derivation.

Function `as.der()` returns a derivation with `as.der(f)(g)=fg-gf`.

Usage

```
as.der(S)
```

Arguments

`S`	Weyl object

Value

Returns a function, a derivation

Author(s)

Robin K. S. Hankin

Examples

```
(o <- rweyl(n=2,d=2))
(f <- as.der(o))

d1 <-rweyl(n=1,d=2)
d2 <-rweyl(n=2,d=2)

f(d1*d2) == d1*f(d2) + f(d1)*d2 # should be TRUE
```

`dim` *The dimension of a* `weyl` *object*

Description

The *dimension* of a weyl algebra is the number of variables needed; it is half the `spray::arity()`. The *dimension* of a Weyl algebra generated by $\{x_1, x_2, \ldots, x_n, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$ is n (not $2n$).

Usage

```
## S3 method for class 'weyl'
dim(x)
```

Arguments

`x`	Object of class `weyl`

Value

Integer

Note

Empty spray objects give zero-dimensional weyl objects.

Author(s)

Robin K. S. Hankin

Examples

```
(a <- rweyl())
dim(a)
```

`dot-class` *Class "dot"*

Description

The dot object is defined so that idiom like `.[x,y]` returns the commutator, that is, `xy-yx`.

The dot object is generated by running script `inst/dot.Rmd`, which includes some further discussion and technical documentation, and creates file `dot.rda` which resides in the `data/` directory.

Arguments

`x`	Object of any class
`i,j`	elements to commute
`...`	Further arguments to `dot_error()`, currently ignored

Value

Always returns an object of the same class as `xy`.

Author(s)

Robin K. S. Hankin

Examples

```
x <- rweyl(n=1,d=2)
y <- rweyl(n=1,d=2)
z <- rweyl(n=1,d=2)

.[x,.[y,z]] + .[y,.[z,x]] + .[z,.[x,y]]  # Jacobi identity
```

`drop` *Drop redundant information*

Description

Coerce constant weyl objects to numeric

Usage

```
drop(x)
```

Arguments

`x`	Weyl object

Details

If its argument is a constant weyl object, coerce to numeric.

Value

Returns either a length-one numeric vector or its argument, a weyl object

Note

Many functions in the package take `drop` as an argument which, if `TRUE`, means that the function returns a `drop`ped value.

Author(s)

Robin K. S. Hankin

Examples

```
a <- rweyl() + 67
drop(a)

drop(idweyl(9))

drop(constant(a,drop=FALSE))
```

grade — *The grade of a weyl object*

Description

The *grade* of a homogeneous term of a Weyl algebra is the sum of the powers. Thus the grade of $4xy^2\partial_x^3\partial_y^4$ is $1+2+3+4=10$.

The functionality documented here closely follows the equivalent in the **clifford** package.

Coutinho calls this the *symbol map*.

Usage

```
grade(C, n, drop=TRUE)
grade(C,n) <- value
grades(x)
```

Arguments

`C,x`	Weyl object
`n`	Integer vector specifying grades to extract
`value`	Replacement value, a numeric vector
`drop`	Boolean, with default `TRUE` meaning to coerce a constant operator to numeric, and `FALSE` meaning not to

Details

Function `grades()` returns an (unordered) vector specifying the grades of the constituent terms. Function `grades<-()` allows idiom such as `grade(x,1:2) <- 7` to operate as expected [here to set all coefficients of terms with grades 1 or 2 to value 7].

Function `grade(C,n)` returns a Weyl object with just the elements of grade g, where `g %in% n`.

The zero grade term, `grade(C,0)`, is given more naturally by `constant(C)`.

Value

Integer vector or weyl object

Author(s)

Robin K. S. Hankin

Examples

```
a <- rweyl(30)

grades(a)
grade(a,1:4)
grade(a,5:9) <- -99
a
```

`identity` *The identity operator*

Description

The identity operator maps any function to itself.

Usage

```
idweyl(d)
## S3 method for class 'weyl'
as.id(S)
is.id(S)
```

Arguments

`d`	Integer specifying dimensionality of the weyl object (twice the spray arity)
`S`	A weyl object

Value

A weyl object corresponding to the identity operator

Note

The identity function cannot be called "`id()`" because then R would not know whether to create a `spray` or a `weyl` object.

Examples

```
idweyl(7)

a <- rweyl(d=5)
a
is.id(a)
is.id(1+a-a)
as.id(a)

a == a*1
a == a*as.id(a)
```

Ops — *Arithmetic Ops Group Methods for the Weyl algebra*

Description

Allows arithmetic operators to be used for spray calculations, such as addition, multiplication, division, integer powers, etc.

Idiom such as `x^2 + y*z/5` should work as expected. Operations are the same as those of the **spray** package except for `*`, which is interpreted as functional composition. A number of helper functions are documented here (which are not designed for the end-user).

Usage

```
## S3 method for class 'weyl'
Ops(e1, e2 = NULL)
weyl_prod_helper1(a,b,c,d)
weyl_prod_helper2(a,b,c,d)
weyl_prod_helper3(a,b,c,d)
weyl_prod_univariate_onerow(S1,S2,func)
weyl_prod_univariate_nrow(S1,S2)
weyl_prod_multivariate_onerow_singlecolumn(S1,S2,column)
weyl_prod_multivariate_onerow_allcolumns(S1,S2)
weyl_prod_multivariate_nrow_allcolumns(S1,S2)
weyl_power_scalar(S,n)
```

Arguments

`S,S1,S2,e1,e2`	Objects of class `weyl`, elements of a Weyl algebra
`a,b,c,d`	Integers, see details
`column`	column to be multiplied
`n`	Integer power (non-negative)
`func`	Function used for products

Details

All arithmetic is as for `spray` objects, apart from `*` and `^`. Here, `*` is interpreted as operator concatenation: Thus, if w_1 and w_2 are Weyl objects, then $w_1 w_2$ is defined as the operator that takes f to $w_1(w_2 f)$.

Functions such as `weyl_prod_multivariate_nrow_allcolumns()` are low-level helper functions with self-explanatory names. In this context, "`univariate`" means the first Weyl algebra, generated by $\{x, \partial\}$, subject to $x\partial - \partial x = 1$; and "`multivariate`" means the algebra generated by $\{x_1, x_2, \ldots, x_n, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$.

The product is somewhat user-customisable via option `prodfunc`, which affects function `weyl_prod_univariate_onero`
Currently the package offers three examples: `weyl_prod_helper1()`, `weyl_prod_helper2()`, and `weyl_prod_helper3()`. These are algebraically identical but occupy different positions on the efficiency-readability scale. The option defaults to `weyl_prod_helper3()`, which is the fastest but most opaque. The vignette provides further details, motivation, and examples.

Value

Generally, return a weyl object

Note

Function `weyl_prod_univariate_nrow()` is present for completeness, it is not used in the package

Author(s)

Robin K. S. Hankin

Examples

```
x <- rweyl(n=1,d=2)
y <- rweyl(n=1,d=2)
z <- rweyl(n=2,d=2)


x*(y+z) == x*y + x*z
is.zero(x*(y*z) - (x*y)*z)
```

`print.weyl` *Print methods for weyl objects*

Description

Printing methods for weyl objects follow those for the **spray** package, with some additional functionality.

Usage

```
## S3 method for class 'weyl'
print(x, ...)
```

Arguments

`x`	A weyl object
`...`	Further arguments, currently ignored

Details

Option `polyform` determines whether the object is to be printed in matrix form or polynomial form: as in the **spray** package, this option governs dispatch to either `print_spray_polyform()` or `print_spray_matrixform()`.

```
> a <- rweyl()
> a    # default print method
A member of the Weyl algebra:
  x  y  z dx dy dz     val
  1  2  2  2  1  0  =    3
  2  2  0  0  1  1  =    2
```

```
  0  0  0  1  1  2  =    1
> options(polyform = TRUE)
> a
A member of the Weyl algebra:
+3*x*y^2*z^2*dx^2*dy +2*x^2*y^2*dy*dz +dx*dy*dz^2
> options(polyform = FALSE)  # restore default
```

Irrespective of the value of `polyform`, option `weylvars` controls the variable names. If `NULL` (the default), then sensible values are used: either `[xyz]` if the dimension is three or less, or integers. But option `weylvars` is user-settable:

```
> options(weylvars=letters[18:20])
> a
A member of the Weyl algebra:
  r  s  t dr ds dt     val
  1  2  2  2  1  0  =    3
  2  2  0  0  1  1  =    2
  0  0  0  1  1  2  =    1
> options(polyform=TRUE)
> a
A member of the Weyl algebra:
+3*r*s^2*t^2*dr^2*ds +2*r^2*s^2*ds*dt +dr*ds*dt^2
> options(polyform=FALSE) ; options(weylvars=NULL)
```

If the user sets `weylvars`, the print method tries to do the Right Thing (tm). If set to `c("a","b","c")`, for example, the generators are named `c(" a"," b"," c","da","db","dc")` [note the spaces]. If the algebra is univariate, the names will be something like `d` and `x`. No checking is performed and if the length is not equal to the dimension, undesirable behaviour may occur. For the love of God, do not use a variable named `d`. Internally, `weylvars` works by changing the `sprayvars` option in the **spray** package.

Note that, as for `spray` objects, this option has no algebraic significance: it only affects the print method.

Value

Returns a `weyl` object.

Author(s)

Robin K. S. Hankin

Examples

```
a <- rweyl()
print(a)
options(polyform=TRUE)
print(a)
```

`rweyl` *Random weyl objects*

Description

Creates random weyl objects: quick-and-dirty examples of Weyl algebra elements

Usage

```
rweyl(nterms = 3, vals = seq_len(nterms), dim = 3, powers = 0:2)
```

Arguments

`nterms`	Number of terms in output
`vals`	Values of coefficients
`dim`	Dimension of weyl object
`powers`	Set from which to sample the entries of the index matrix

Value

Returns a weyl object

Author(s)

Robin K. S. Hankin

Examples

```
rweyl()

rweyl(d=7)
```

`spray` *Create spray objects*

Description

Function `spray()` creates a sparse array; function `weyl()` coerces a spray object to a Weyl object.

Usage

```
spray(M,x,addrepeats=FALSE)
```

Arguments

`M`	An integer-valued matrix, the index of the weyl object
`x`	Numeric vector of coefficients
`addrepeats`	Boolean, specifying whether repeated rows are to be added

Details

The function is discussed and motivated in the **spray** package.

Value

Return a weyl or a Boolean

Author(s)

Robin K. S. Hankin

Examples

```
(W <- spray(matrix(1:36,6,6),1:6))
weyl(W)

as.weyl(15,d=3)
```

`weyl` *The algebra and weyl objects*

Description

Basic functions for weyl objects

Usage

```
weyl(M)
is.weyl(M)
as.weyl(val,d)
is.ok.weyl(M)
```

Arguments

`M`	A weyl or spray object
`val,d`	Value and dimension for weyl object

Details

Function `weyl()` is the formal creator method; `is.weyl()` tests for weyl objects and `is.ok.weyl()` checks for well-formed sprays. Function `as.weyl()` tries (but not very hard) to infer what the user intended and return the right thing.

To create a spray object to pass to `weyl()`, use function `spray()`, which is a synonym for `spray::spray()`.

Value

Return a weyl or a Boolean

Author(s)

Robin K. S. Hankin

Examples

```
(W <- spray(matrix(1:36,6,6),1:6))
weyl(W)

as.weyl(15,d=3)
```

---

weyl-class *Class "weyl"*

Description

The formal S4 class for weyls.

Objects from the Class

Objects *can* be created by calls of the form `new("weyl", ...)` but this is not encouraged. Use functions `weyl()` or `as.weyl()` instead.

Author(s)

Robin K. S. Hankin

---

x_and_d *Generating elements for the first Weyl algebra*

Description

Variables `x` and `d` correspond to operator x and ∂_x; they are provided for convenience. These elements generate the one-dimensional Weyl algebra.

Note that a similar system for multivariate Weyl algebras is not desirable. We might want to consider the Weyl algebra generated by $\{x, y, z, \partial_x, \partial_y, \partial_z\}$ and correspondingly define R variables `x,y,z,dx,dy,dz`. But then variable `x` is ambiguous: is it a member of the first Weyl algebra or the third?

Usage

```
data(x_and_d)
```

Author(s)

Robin K. S. Hankin

Examples

```
d
x

.[d,x]   # dx-xd==1

d^3 * x^4

(1-d*x*d)*(x^2-d^3)
```

`zero` *The zero operator*

Description

The zero operator maps any function to the zero function (which maps anything to zero). To test for being zero, use `spray::is.zero()`; package idiom would be `is.zero()`.

Usage

```
zero(d)
```

Arguments

`d`	Integer specifying dimensionality of the weyl object (twice the spray arity)

Value

A weyl object corresponding to the zero operator (or a Boolean for `is.zero()`)

Examples

```
(a <- rweyl(d=5))
is.zero(a)
is.zero(a-a)
is.zero(a*0)

a == a + zero(dim(a))
```

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